

On the harmonic measure and capacity of rational lemniscates

Stamatis Pouliasis · Thomas Ransford

Abstract We study the lemniscates of rational maps. We prove a reflection principle for the harmonic measure of rational lemniscates and we give estimates for their capacity and the capacity of their components. Also, we prove a version of Schwarz's lemma for the capacity of the lemniscates of proper holomorphic functions.

Keywords Rational functions · lemniscates · harmonic measure · logarithmic capacity · Lindelöf principle.

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1 Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc and let R be a rational function in the extended complex plane $\hat{\mathbb{C}}$ with $R(\infty) = 0$. A set of the form

$$\{z \in \hat{\mathbb{C}} : |R(z)| = t\}, \quad 0 < t < \infty,$$

is called a *lemniscate* of R ; we will also refer to sets of the form

$$K_t := \{z \in \hat{\mathbb{C}} : |R(z)| \geq t\}, \quad 0 < t < \infty,$$

as lemniscates of R . The properties of lemniscates of polynomials and rational functions have been studied by many researchers. We mention here some recent results.

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Anderson and Eiderman [2] proved that there exists an absolute constant $C > 0$ such that, for the logarithmic derivative

$$\frac{Q'_n(z)}{Q_n(z)} = \sum_{i=1}^n \frac{1}{z - z_i}$$

of every polynomial $Q_n(z) := \prod_{i=1}^n (z - z_i)$ of degree n , the inequality

$$M\left(\left\{z \in \mathbb{C} : \left|\sum_{i=1}^n \frac{1}{z - z_i}\right| > t\right\}\right) \leq \frac{C}{t} n \sqrt{\log n}$$

holds, where M denotes the 1-dimensional Hausdorff content.

Solynin and Williams [18] proved that, for each $n \geq 1$, there exists a constant $C(n)$ such that the inequality

$$\frac{\lambda(\{z \in \hat{\mathbb{C}} : |P(z)| \leq c\})}{\pi r^2(\{z \in \hat{\mathbb{C}} : |P(z)| \leq c\})} \leq C(n)$$

holds for every complex polynomial P of degree n and for every $c \in (0, +\infty)$, where $\lambda(E)$ is the area of E and $r(E)$ is the inradius of E (i.e. the supremum of the radii of open disks contained in E).

A map between two topological spaces $F : X \rightarrow Y$ is called *proper* if the inverse image $F^{-1}(K)$ of every compact subset K of Y is a compact subset of X . Dubinin [8], among other results, generalized a result of Pólya for the area of a polynomial lemniscate by proving the following inequality for a proper holomorphic map F from a domain D onto a circular ring $\{z \in \mathbb{C} : t_1 < |z| < t_2\}$ ($0 < t_1 < t_2 < +\infty$): if E is the union of all those connected components of $\hat{\mathbb{C}} \setminus D$ whose boundaries contain points corresponding, under the holomorphic function F , to points on the circle $\{z \in \mathbb{C} : |z| = t_1\}$ and $\infty \notin E$, then

$$\left(\frac{t_2}{t_1}\right)^{\frac{2}{n}} \leq \frac{\lambda(E \cup D)}{\lambda(E)}. \quad (1.1)$$

Also, he proved that equality holds in (1.1) if and only if $F(z) = c(z - a)^n$, where c and a are arbitrary complex numbers.

Let Γ be a C^∞ Jordan curve in \mathbb{C} and let G_- and G_+ denote the bounded and unbounded component of $\hat{\mathbb{C}} \setminus \Gamma$ respectively. From the Riemann mapping theorem there exist conformal maps $\phi_- : \mathbb{D} \mapsto G_-$ and $\phi_+ : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \mapsto G_+$ with $\phi_+(\infty) = \infty$ and $\phi'_+(\infty) > 0$. It is well known that ϕ_- , ϕ_+ extend to C^∞ diffeomorphisms on the closures of their respective domains. The map $\phi_+^{-1} \circ \phi_- : \partial\mathbb{D} \mapsto \partial\mathbb{D}$ is called the *fingerprint* of Γ . Ebenfelt, Khavinson and Shapiro [10], among other results, proved that the fingerprint of a polynomial lemniscate of degree n is given by the n -th root of a Blaschke product of degree n and that conversely, any smooth diffeomorphism induced by such a map is the fingerprint of a polynomial lemniscate of the same degree. Younsi [20] generalized the above result to the case of rational lemniscates.

For more results and applications of lemniscates we refer the reader to the books [4], [14] and [16].

The starting point of our work was a question posed by Younsi considering the capacity of the components of the lemniscate of a good rational function. Following [11], we will say that a rational function R is *d-good* ($d \in \mathbb{N}$) if the degree of R is d , if $R(\infty) = 0$ and if the open set $\Omega := R^{-1}(\mathbb{D})$ is connected and bounded by d disjoint analytic Jordan curves γ_i , $i = 1, \dots, d$. Then R has a simple pole p_i on the bounded component of $\hat{\mathbb{C}} \setminus \gamma_i$ for each i , and it can be written as

$$R(z) = \sum_{i=1}^d \frac{a_i}{z - p_i},$$

for some $a_i \in \mathbb{C} \setminus \{0\}$, $i = 1, \dots, d$. Also, we will denote by ζ_1, \dots, ζ_d the zeros of R (repeated according to multiplicity). We prove the following reflection principle for the harmonic measure of Ω and $\hat{\mathbb{C}} \setminus \overline{\Omega}$: for every Borel set $E \subset \partial\Omega$,

$$\sum_{i=1}^d \omega_{p_i}^{\hat{\mathbb{C}} \setminus \overline{\Omega}}(E) = \sum_{j=1}^d \omega_{\zeta_j}^{\Omega}(E),$$

where ω_a^D denotes the harmonic measure of an open set $D \subset \hat{\mathbb{C}}$ with respect to the point $a \in D$ (the above equality is true for arbitrary rational functions, see Theorem 3.1). Also, we show that the above equality characterizes rational functions in the class of proper holomorphic functions (see Theorem 3.2). For the logarithmic capacity of the component K_i of $K := \hat{\mathbb{C}} \setminus \Omega$ containing p_i , we give a new proof of the known result that

$$\text{cap}(K_i) \geq |a_i|, \quad i = 1, \dots, d,$$

and we show that there exists a constant c , depending just on the radius of injectivity of R on K_i , such that

$$\text{cap}(K_i) \leq c|a_i|.$$

From [19, Proposition 4.16, p. 114] it follows that there exists an absolute constant $C > 0$ such that, for the lemniscate $K := \{z \in \mathbb{C} : |R(z)| \geq 1\}$ of every good rational function $R(z) := \sum_{i=1}^d (a_i/(z - p_i))$,

$$\gamma(K) \leq C \sum_{i=1}^d |a_i|, \quad (1.2)$$

where γ denotes analytic capacity. Younsi, motivated by considerations related to the semi-additivity property of analytic capacity, asked the following question:

Question 1.1. *Given $d \geq 2$, does there exist a constant $C(d) > 0$ with the following property: if $R(z) := \sum_{i=1}^d (a_i/(z - p_i))$ is a d -good rational function, then*

$$\text{cap}(K_i) \leq C(d)|a_i|,$$

where K_i is the component of the lemniscate $K := \{z \in \mathbb{C} : |R(z)| \geq 1\}$ containing p_i ?

We answer negatively Younsi's question by giving examples of good rational functions of degree 3 such that the ratio $\text{cap}(K_i)/|a_i|$ can be arbitrarily large. It is well known that, if $P(z) := \sum_{i=0}^n a_i z^i$ is a polynomial with $a_n \neq 0$ and E is a compact subset of \mathbb{C} , then the logarithmic capacity of $P^{-1}(E)$ is given by

$$\text{cap}(P^{-1}(E)) = \left(\frac{\text{cap}(E)}{|a_n|} \right)^{\frac{1}{n}},$$

see [15, Theorem 5.2.5, p. 134]. We give a lower estimate for the logarithmic capacity of the lemniscate K of a good rational function R taking into account the poles $\{p_i\}$ and the residues $\{a_i\}$ of R by showing that

$$\text{cap}(K) \geq \left[\prod_{\substack{i,j=1 \\ i \neq j}}^d |p_i - p_j| \prod_{i=1}^d |a_i| \right]^{\frac{1}{d^2}},$$

(see Theorems 4.1, 4.4 and 4.6).

Finally, if f is a proper holomorphic function from a domain Ω to \mathbb{D} with $f(\infty) = 0$, we prove a geometric version of Schwarz's lemma for the logarithmic capacity of the lemniscates $K_t := \hat{\mathbb{C}} \setminus \{z \in \Omega : |f(z)| < t\}$ by showing that the function

$$t \mapsto t^{\frac{1}{m(\infty)}} \cdot \text{cap}(K_t), \quad t \in (0, 1),$$

where $m(\infty)$ is the multiplicity of f at ∞ , is non-decreasing and it is constant on a neighborhood of 0 (see Theorem 4.2).

2 Notations and preliminaries

2.1 Good rational functions

Let $R(z) := \sum_{i=1}^d (a_i/(z - p_i))$ be a d -good rational function and let K_i be the component of the lemniscate $K := \hat{\mathbb{C}} \setminus \Omega$ containing p_i . Then R is injective on a neighborhood V_i of K_i and we will denote by R_i the restriction of R on V_i , $i = 1, \dots, d$. We will denote by D_i the interior of K_i . Also, we let $Q_i := 1/R_i$ on V_i and $P_i := Q_i^{-1}$, and we note that $P_i : \mathbb{D} \mapsto K_i$ is one to one and onto, $i = 1, \dots, d$.

2.2 Harmonic measure and logarithmic capacity

We will denote by $G_D(z, a)$, $z \in D$, and $\omega_a^D(E)$, $E \subset \partial D$, the *Green function* and the *harmonic measure* of a Greenian domain $D \subset \hat{\mathbb{C}}$ with respect to the point $a \in D$. (Here *Greenian* means simply that the domain possesses a Green function. It is well known that a planar domain is Greenian if and only if its complement is of positive logarithmic capacity.) Also, we let $G_D(z, a) := 0$ for $z \in \hat{\mathbb{C}} \setminus D$ and we note that, for domains D that are regular for the Dirichlet problem, $z \mapsto G_D(z, a)$ is a subharmonic function on $\hat{\mathbb{C}} \setminus \{a\}$.

The *equilibrium energy* of a compact set $K \subset \mathbb{C}$ is defined by

$$I(K) := \inf_{\mu} \iint \log \frac{1}{|z - w|} d\mu(z) d\mu(w),$$

where the infimum is taken over all Borel probability measures μ supported on K . When $I(K) < +\infty$ the above infimum is attained by a unique probability measure μ_K supported on ∂K , which is called the *equilibrium measure* of K . The *logarithmic capacity* of K is defined by

$$\text{cap}(K) := e^{-I(K)}.$$

The logarithmic capacity is related to the Green function by the following formula ([15, Theorem 5.2.1 p. 132])

$$\text{cap}(K) = \exp \left(- \lim_{z \rightarrow \infty} (G_{\hat{\mathbb{C}} \setminus K}(z, \infty) - \log |z|) \right). \quad (2.1)$$

For more information about potential theory in the complex plane see e.g. [15].

2.3 Lindelöf principle

Let f be a non-constant holomorphic function on a Greenian domain D such that $f(D)$ is Greenian. The following inequality is known as the *Lindelöf principle*:

$$G_{f(D)}(w_0, f(z)) \geq \sum_{f(a)=w_0} m(a) G_D(a, z),$$

where $z \in D$, $w_0 \in f(D)$ and $m(a)$ is the *multiplicity* of the zero of $f(z) - f(a)$ at $a \in D$. It is well known that, if f is a proper holomorphic function from D to $f(D)$, then equality holds in the Lindelöf principle (see e.g. [12]). For a characterization of the equality cases in the Lindelöf principle see [3].

2.4 A majorization principle for harmonic measure under meromorphic functions

We will use the following result of Dubinin for the behavior of harmonic measure under certain meromorphic functions.

Theorem 2.1 ([7, Theorem 2, p.753]). *Let D and G be domains bounded by finitely many Jordan curves and let f be a meromorphic function on D such that $f(\partial D) \subset \mathbb{C} \setminus G$. Suppose that the sets $\gamma \subset \partial D$ and $f(\gamma) \subset \partial G$ consist of finitely many open arcs, and positively oriented arcs from γ are mapped by f to positively oriented arcs on ∂G . If $w_0 \in f(D)$, then*

$$\omega_{w_0}^G(f(\gamma)) \leq \sum_{i=1}^m \omega_{z_i}^D(\gamma), \quad (2.2)$$

where z_1, \dots, z_m are the zeros of the function $f - w_0$ if $w_0 \neq \infty$ and the zeros of $1/f$ if $w_0 = \infty$ with multiplicities taken into account. Equality in (2.2) is attained if and only if f is a proper meromorphic function from D to G and the map $f : \gamma \mapsto f(\gamma)$ is one to one.

We note that inequality (2.2) and the equality statement of Theorem 2.1 remain true if we replace γ with an arbitrary Borel set $E \subset \gamma$.

3 A reflection principle for harmonic measure

In the following theorem we prove a reflection principle for the harmonic measure of rational lemniscates, taking into account the zeros and the poles of the rational function.

Theorem 3.1. *Let R be a rational function of degree d , let ζ_1, \dots, ζ_d be the zeros and p_1, \dots, p_d be the poles of R and let $\Omega := R^{-1}(\mathbb{D})$. Then*

$$\sum_{i=1}^d \omega_{p_i}^{\hat{\mathbb{C}} \setminus \overline{\Omega}}(E) = \sum_{j=1}^d \omega_{\zeta_j}^{\Omega}(E), \quad (3.1)$$

for every Borel set $E \subset \partial\Omega$.

Proof. Let $\Gamma := \partial\Omega \setminus \{\zeta \in \partial\Omega : R'(\zeta) = 0\}$. Then there is a decomposition of Γ by half-open arcs $\Gamma_1, \dots, \Gamma_d$ such that R is injective on Γ_i , $i = 1, \dots, d$. Fix $i \in \{1, 2, \dots, d\}$ and let E be a Borel subset of Γ_i . Let A_1 be the connected component of Ω with $E \subset \partial A_1$ and let B_1 be the set of zeros of R on A_1 . Also, let A_2 be the connected component of $\hat{\mathbb{C}} \setminus \overline{\Omega}$ with $E \subset \partial A_2$

and let B_2 be the set of poles of R on A_2 . We note that R is a proper meromorphic function from A_1 to \mathbb{D} and from A_2 to $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. From Theorem 2.1,

$$\begin{aligned} \sum_{i=1}^d \omega_{p_i}^{\hat{\mathbb{C}} \setminus \overline{\Omega}}(E) &= \sum_{p \in B_2} m(p) \omega_p^{\hat{\mathbb{C}} \setminus \overline{\Omega}}(E) = \omega_{\infty}^{\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}}(R(E)) \\ &= \omega_0^{\mathbb{D}}(R(E)) = \sum_{\zeta \in B_1} m(\zeta) \omega_{\zeta}^{\Omega}(E) = \sum_{j=1}^d \omega_{\zeta_j}^{\Omega}(E). \end{aligned} \quad (3.2)$$

For an arbitrary Borel set $E \subset \partial\Omega$, we may assume that $E \subset \Gamma$, since harmonic measure does not change by removing a finite number of points from E . Then, from the equality (3.2),

$$\sum_{j=1}^d \omega_{\zeta_j}^{\Omega}(E) = \sum_{n=1}^d \sum_{j=1}^d \omega_{\zeta_j}^{\Omega}(E \cap \Gamma_n) = \sum_{n=1}^d \sum_{i=1}^d \omega_{p_i}^{\hat{\mathbb{C}} \setminus \overline{\Omega}}(E \cap \Gamma_n) = \sum_{i=1}^d \omega_{p_i}^{\hat{\mathbb{C}} \setminus \overline{\Omega}}(E). \quad \square$$

Remark. Theorem 3.1 is a close relative of problems on the proportionality of harmonic measures studied in [1] and [17].

In the next theorem we show that the reflection principle for rational lemniscates proved above actually characterizes rational functions among proper holomorphic functions.

Theorem 3.2. *Let Ω be a finitely connected domain bounded by d disjoint analytic Jordan curves $\gamma_1, \dots, \gamma_d$, with $\infty \in \Omega$. Let f be a proper holomorphic function of degree d from Ω to \mathbb{D} and let ζ_1, \dots, ζ_d be its zeros. Suppose further that, for every $i = 1, \dots, d$ there exists p_i in the bounded component of $\hat{\mathbb{C}} \setminus \gamma_i$ such that*

$$\sum_{i=1}^d \omega_{p_i}^{\hat{\mathbb{C}} \setminus \overline{\Omega}}(E) = \sum_{j=1}^d \omega_{\zeta_j}^{\Omega}(E) \quad (3.3)$$

for every Borel set $E \subset \partial\Omega$. Then f is a rational function.

Proof. From the translation-invariance of harmonic measure we may assume that $f(\infty) = 0$. Let D_i be the bounded component of $\hat{\mathbb{C}} \setminus \gamma_i$, $i = 1, \dots, d$. We note that f has an analytic continuation on a neighborhood V_i of ∂D_i and we may choose it such that the restriction f_i of f on V_i is injective, $i = 1, \dots, d$. Suppose that (3.3) holds and let m be the normalized Lebesgue measure on the circle $\partial\mathbb{D}$. From Theorem 2.1,

$$\sum_{j=1}^d \omega_{\zeta_j}^{\Omega} = \omega_0^{\mathbb{D}} \circ f_i = m \circ f_i, \quad \text{on } V_i,$$

for every $i = 1, \dots, d$. From (3.3) we have

$$\sum_{j=1}^d \omega_{\zeta_j}^{\Omega} = \omega_{p_i}^{D_i}, \quad \text{on } V_i.$$

Therefore $m \circ f_i = \omega_{p_i}^{D_i}$ on V_i , $i = 1, \dots, d$. Let ϕ_i be a conformal map of D_i onto $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ with $\phi_i(p_i) = \infty$. We may assume that ϕ_i has an analytic continuation on V_i . From the conformal invariance of harmonic measure we obtain that $\omega_{p_i}^{D_i} = \omega_{\infty}^{\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}} \circ \phi_i = m \circ \phi_i$. Therefore $m \circ f_i = m \circ \phi_i$ or $m \circ (f_i(\phi_i^{-1}(E))) = m(E)$, for every Borel set $E \subset \partial\mathbb{D}$. We obtain that $f_i \circ \phi_i^{-1}$ is a diffeomorphism of $\partial\mathbb{D}$ that preserves Lebesgue measure. Therefore, $f_i \circ \phi_i^{-1}(\zeta) = \lambda \zeta$, or

$f_i(\zeta) = \lambda \phi_i(\zeta)$ for every $\zeta \in \partial\mathbb{D}$ and for some $\lambda \in \partial\mathbb{D}$. From the identity principle we have that $f_i(\zeta) = \lambda \phi_i(\zeta)$ for every $\zeta \in V_i$. Since this is true for every $i = 1, \dots, d$, we obtain an extension of f as a meromorphic function on $\hat{\mathbb{C}}$ with poles at p_1, \dots, p_d . Therefore, f is a rational function. \square

4 Capacity of rational lemniscates

In the following theorem we consider the logarithmic capacity of the lemniscate K of a d -good rational function R and of its components K_i , $i = 1, \dots, d$.

Theorem 4.1. *Let R be a d -good rational function. Then*

$$\text{cap}(K_i) \geq |a_i|, \quad i = 1, \dots, d, \quad (4.1)$$

and

$$\text{cap}(K) \geq \left[\prod_{\substack{i,j=1 \\ i \neq j}}^d |p_i - p_j| \prod_{i=1}^d |a_i| \right]^{\frac{1}{d^2}}. \quad (4.2)$$

Remark. Inequality (4.1) is a form of Lavrent'ev's inequality on the product of conformal radii of two non-overlapping simply connected domains (see e.g. [13, p.223, Corollary 1] or [6]). The proof given below is different. We believe that inequality (4.2) is new, though several similar inequalities were obtained in [13, Chapter 3, §6].

Proof. Let D_i denote the interior of K_i , $i = 1, \dots, d$. Let $\mu := \sum_{i=1}^d \omega_{p_i}^{D_i}$. We have $I(\mu) = \sum_{i,j=1}^d I(\omega_{p_i}^{D_i}, \omega_{p_j}^{D_j})$. For $i \neq j$,

$$\begin{aligned} I(\omega_{p_i}^{D_i}, \omega_{p_j}^{D_j}) &= \iint \log \frac{1}{|z-w|} d\omega_{p_i}^{D_i}(z) d\omega_{p_j}^{D_j}(w) \\ &= \int \log \frac{1}{|p_i-w|} d\omega_{p_j}^{D_j}(w) \\ &= \log \frac{1}{|p_i-p_j|}. \end{aligned}$$

For $i = j$, we note that

$$H_{D_i}(z, w) := G_{D_i}(z, w) - \log \frac{1}{|z-w|} = - \int \log \frac{1}{|a-w|} d\omega_z^{D_i}(a)$$

since D_i is bounded ([15, Theorem 4.4.7, p. 110]), and we obtain

$$\begin{aligned}
I(\omega_{p_i}^{D_i}) &= \int \int \log \frac{1}{|z-w|} d\omega_{p_i}^{D_i}(z) d\omega_{p_i}^{D_i}(w) \\
&= - \int H_{D_i}(p_i, w) d\omega_{p_i}^{D_i}(w) \\
&= -H_{D_i}(p_i, p_i) \\
&= - \lim_{z \rightarrow p_i} \left[G_{D_i}(z, p_i) - \log \frac{1}{|z-p_i|} \right] \\
&= - \lim_{z \rightarrow p_i} \left[G_{\hat{\mathbb{C}} \setminus \overline{D}}(R_i(z), \infty) - \log \frac{1}{|z-p_i|} \right] \\
&= - \lim_{z \rightarrow p_i} \log |(z-p_i)R_i(z)| \\
&= - \lim_{z \rightarrow p_i} \log \left| \sum_{j=1}^d \frac{a_j(z-p_i)}{z-p_j} \right| \\
&= \log \frac{1}{|a_i|}.
\end{aligned}$$

Therefore,

$$I(\mu) = \sum_{\substack{i,j=1 \\ i \neq j}}^d \log \frac{1}{|p_i - p_j|} + \sum_{i=1}^d \log \frac{1}{|a_i|}. \quad (4.3)$$

Inequality (4.1) follows from

$$I(K_i) \leq I(\omega_{p_i}^{D_i}) = \log \frac{1}{|a_i|}$$

and inequality (4.2) follows from

$$I(K) \leq I\left(\frac{\mu}{d}\right) = \frac{1}{d^2} I(\mu),$$

since μ/d is a Borel probability measure on K . \square

Remark. Let R be a rational function of degree d having simple poles p_i , $i = 1, \dots, d$, and satisfying $R(\infty) = 0$. Let a_i be the residue of R at p_i , $i = 1, \dots, d$. Then, there exists $t_0 \in (0, +\infty)$ such that, for every $t \in (t_0, +\infty)$, $z \mapsto R(z)/t$ is a d -good rational function. Let $K_{i,t}$ be the component of the lemniscate $K_t := \{z \in \hat{\mathbb{C}} : |R(z)| \geq t\}$ containing p_i , $t \in (t_0, +\infty)$. Then, from Theorem 4.1, we obtain that

$$t \cdot \text{cap}(K_{i,t}) \geq |a_i|, \quad i = 1, \dots, d,$$

and

$$t^{\frac{1}{d}} \cdot \text{cap}(K_t) \geq \left[\prod_{\substack{i,j=1 \\ i \neq j}}^d |p_i - p_j| \prod_{i=1}^d |a_i| \right]^{\frac{1}{d^2}},$$

for every $t \in (t_0, +\infty)$.

We will also examine the behavior of the logarithmic capacity of the lemniscates K_t for $t \in (0, t_0)$. In fact, we will consider proper holomorphic functions. Burckel, Marshall, Minda, Poggi-Corradini and Ransford [5] proved geometric versions of Schwarz's lemma for a holomorphic function f on the unit disc \mathbb{D} by showing that the function

$$r \mapsto \frac{T(f(r\mathbb{D}))}{T(r\mathbb{D})}, \quad 0 < r < 1,$$

is increasing, where $T(E)$ may be area, diameter or logarithmic capacity of E . In the same article they asked about analogues of Schwarz's lemma for the dual situation of holomorphic functions defined on a domain Ω onto the unit disc, where Ω satisfies some geometric restriction. Dubinin's inequality (1.1) is a result of this type considering the area of the lemniscates of a proper holomorphic function from a domain in $\hat{\mathbb{C}}$ to a circular ring. In the following theorem we prove a monotonicity principle for the logarithmic capacity of the lemniscates of a proper holomorphic function from a finitely connected domain to the unit disc.

Theorem 4.2. *Let f be a proper holomorphic function from a domain $\Omega \subset \hat{\mathbb{C}}$ to \mathbb{D} such that $\infty \in \Omega$ and $f(\infty) = 0$. For every $t \in (0, 1)$ we let*

$$\Omega_t := \{z \in \Omega : |f(z)| < t\}$$

and $K_t := \hat{\mathbb{C}} \setminus \Omega_t$. Then the function

$$F(t) := t^{\frac{1}{m(\infty)}} \cdot \text{cap}(K_t), \quad t \in (0, 1),$$

is non-decreasing and there exists $t_1 \in (0, 1)$ such that

$$F(t) = |f^{(m(\infty))}(\infty)|^{\frac{1}{m(\infty)}}, \quad t \in (0, t_1). \quad (4.4)$$

Proof. For $t \in (0, 1)$, let D_t denote the connected component of Ω_t that contains ∞ , and let $Z(t)$ be the set of zeros of f in $D_t \setminus \{\infty\}$. Note that f is a proper holomorphic function from D_t to \mathbb{D} and that $Z(t_1) \subset Z(t_2)$ for $t_1 \leq t_2$. Also, since the logarithmic capacity of a compact set is equal to the logarithmic capacity of its outer boundary, we have that $\text{cap}(K_t) = \text{cap}(\hat{\mathbb{C}} \setminus D_t)$. From the Lindelöf principle we have that, for every $z \in D_t \setminus (Z(t) \cup \{\infty\})$,

$$\log \frac{t}{|z^{m(\infty)} f(z)|} = m(\infty)(G_{D_t}(z, \infty) - \log |z|) + \sum_{a \in Z(t)} m(a) G_{D_t}(z, a),$$

and letting $z \rightarrow \infty$ we obtain

$$\text{cap}(K_t) = \left(\frac{|f^{(m(\infty))}(\infty)|}{t} \right)^{\frac{1}{m(\infty)}} \exp \left(\sum_{a \in Z(t)} \frac{m(a)}{m(\infty)} G_{D_t}(a, \infty) \right).$$

Then the monotonicity of the function F follows from the positivity and the monotonicity of the Green function and the monotonicity of the sets $Z(t)$. Also, equality (4.4) follows from the fact that there exists $t_1 \in (0, 1)$ such that $Z(t) = \emptyset$ for every $t \in (0, t_1)$. \square

We will make use of the fact that the restriction of a good rational function on one of the components of its lemniscate is univalent. In this direction, the following well-known theorem about the growth and the distortion of univalent functions in the unit disc will be useful.

Theorem 4.3 ([9, Theorem 2.6, p. 33 and Corollary 7, p. 127]). *If f is holomorphic and univalent on \mathbb{D} such that $f(0) = 0$ and $f'(0) = 1$, then*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2},$$

for $|z| = r < 1$ and

$$\frac{1-r^2}{r^2} |f(z)f(w)| \leq \left| \frac{f(z)-f(w)}{z-w} \right| \leq \frac{|f(z)f(w)|}{r^2(1-r^2)},$$

for $|z| = |w| = r < 1$ (for $z = w$ the difference quotient is to be interpreted as $f'(z)$).

Using the above theorem, we obtain the following estimate for the logarithmic capacity of the component K_i of the lemniscate of a good rational function R with respect to the modulus of the corresponding residue a_i of R at p_i , under an injectivity assumption for R on a neighborhood of K_i .

Theorem 4.4. *Let R be a d -good rational function and suppose that*

$$\{z \in \hat{\mathbb{C}} : |z| \geq \frac{1}{r}\} \subset R(V_i)$$

for some $r > 1$, where V_i is a neighborhood of K_i and R is injective on V_i , $i \in \{1, \dots, d\}$. Then

$$\text{cap}(K_i) \leq \frac{r^6}{(r^2-1)(r-1)^4} |a_i|.$$

Proof. We note that $P_i = Q_i^{-1}$ is univalent on $\overline{D(0, r)}$, that $P_i(0) = p_i$ and that $P_i'(0) = a_i$. Let

$$M_i := \sup_{z, w \in \partial K_i} \frac{|z-w|}{|Q_i(z)-Q_i(w)|} = \sup_{z, w \in \partial \mathbb{D}} \frac{|P_i(z)-P_i(w)|}{|z-w|}.$$

From the assumption for r it follows that the function

$$F_i(z) := \frac{P_i(rz) - p_i}{ra_i}$$

is univalent in \mathbb{D} with $F_i(0) = 0$ and $F_i'(0) = 1$. From Theorem 4.3 we have

$$\left| \frac{F_i(z) - F_i(w)}{z-w} \right| \leq \frac{r^4 |F_i(z)F_i(w)|}{r^2-1} \leq \frac{r^6}{(r^2-1)(r-1)^4},$$

which implies that

$$\frac{|P_i(rz) - P_i(rw)|}{|rz - rw|} \leq \frac{r^6}{(r^2-1)(r-1)^4} |a_i|$$

for $|z| = |w| = \frac{1}{r} < 1$. Therefore,

$$M_i \leq \frac{r^6}{(r^2-1)(r-1)^4} |a_i|.$$

Let μ_i be the equilibrium measure of K_i and consider the measure

$$v_i(E) := \mu_i(Q_i^{-1}(E)), \quad E \subset \partial \mathbb{D}.$$

Then,

$$\begin{aligned}
0 = I(\overline{\mathbb{D}}) &\leq I(v_i) \\
&= \int \int \log \frac{1}{|z-w|} dv_i(z) dv_i(w) \\
&= \int \int \log \frac{1}{|Q_i(z) - Q_i(w)|} d\mu_i(z) d\mu_i(w) \\
&\leq \int \int \log \frac{M_i}{|z-w|} d\mu_i(z) d\mu_i(w) \\
&= \log M_i + I(K_i).
\end{aligned}$$

Therefore,

$$\text{cap}(K_i) \leq M_i \leq \frac{r^6}{(r^2-1)(r-1)^4} |a_i|. \quad \square$$

Let R be a good rational function, let $p \in R^{-1}(\mathbb{D})$ and consider the rational function

$$R_\varepsilon(z) := R(z) + \frac{\varepsilon}{z-p},$$

having residue $\varepsilon > 0$ at the extra pole p . As a corollary of Theorem 4.4 we obtain an estimate for the rate of decrease of the logarithmic capacity of the component of the lemniscate of R_ε that contains p , as $\varepsilon \rightarrow 0$.

Corollary 4.5. *Let R be a d -good rational function, let $p \in R^{-1}(\mathbb{D})$ and let*

$$R_\varepsilon(z) := R(z) + \frac{\varepsilon}{z-p}, \quad \varepsilon > 0, z \in \hat{\mathbb{C}}.$$

If K_ε is the component of the lemniscate $\{z \in \hat{\mathbb{C}} : |R_\varepsilon(z)| \geq 1\}$ of R_ε that contains p , then

$$\text{cap}(K_\varepsilon) = \mathcal{O}(\varepsilon), \quad \text{as } \varepsilon \rightarrow 0.$$

Proof. We will denote by K_i , $i = 1, \dots, d$, the components of the lemniscate

$$K := \{z \in \hat{\mathbb{C}} : |R(z)| \geq 1\}$$

of R . Let $\bar{D}(\infty, r) := \{z \in \hat{\mathbb{C}} : |z| \geq 1/r\}$, $r > 0$. Since R is a d -good rational function, there exist $r > 1$ and a neighborhood V_i of K_i , $i = 1, \dots, d$, such that

$$\bar{D}(\infty, 2r) \subset \bigcap_{i=1}^d R(V_i).$$

Since

$$|R_\varepsilon(z) - R(z)| = \frac{\varepsilon}{|z-p|},$$

R_ε converges locally uniformly to R on $\mathbb{C} \setminus \{p\}$ as $\varepsilon \rightarrow 0$. Therefore, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$, the rational function R_ε is $(d+1)$ -good and

$$\bar{D}(\infty, r) \subset \bigcap_{i=1}^d R_\varepsilon(V_i).$$

Since R_ε is a proper holomorphic function from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$ of degree $(d+1)$, for every $\varepsilon < \varepsilon_0$ there exists a neighborhood V_ε of K_ε such that $\bar{D}(\infty, r) \subset R_\varepsilon(V_\varepsilon)$ and R_ε is injective on $R_\varepsilon^{-1}(\bar{D}(\infty, r)) \cap V_\varepsilon$. From Theorem 4.4 we obtain that, for every $\varepsilon < \varepsilon_0$,

$$\text{cap}(K_\varepsilon) \leq \frac{r^6}{(r^2-1)(r-1)^4} \varepsilon$$

and the conclusion follows. \square

Based on the previous results, one may ask if, given $d \geq 2$, there exists a constant $C(d) > 0$ such that $\text{cap}(K_i) \leq C(d)|a_i|$, for every d -good rational function. In the following theorem we show that the answer is no.

Theorem 4.6. *Let $a > 0$ and $\eta \in (\frac{2}{3}, 1)$. For $p > 1$ define*

$$R_p(z) := \frac{a}{z-p} + \frac{p-p^\eta}{z-ip} + \frac{p-p^\eta}{z+ip}.$$

Then there exists $p_0 := p_0(a, \eta)$ such that, for all $p > p_0$,

- (i) R_p is a 3-good rational function,
- (ii) *the component of the lemniscate $\{z \in \hat{\mathbb{C}} : |R_p(z)| \geq 1\}$ containing p has logarithmic capacity at least $ap^{1-\eta}/8$.*

Proof. (i) To show that $\{z \in \hat{\mathbb{C}} : |R_p(z)| \geq 1\}$ has 3 components, it suffices to show that each critical point c of R_p satisfies $|R_p(c)| < 1$ (see [11, Lemma 2.1]). We shall show that this is the case for all p large enough.

The critical points of R_p are the solutions c of

$$\frac{a}{(c-p)^2} + \frac{p-p^\eta}{(c-ip)^2} + \frac{p-p^\eta}{(c+ip)^2} = 0.$$

Simplifying, we obtain

$$(c-p)^3(c+p) = -\frac{a}{2(p-p^\eta)}(c^2+p^2)^2.$$

This has four roots (as expected), namely

$$c = -p + \mathcal{O}(1) \quad \text{and} \quad c = p - a^{1/3}\omega p^{2/3} + \mathcal{O}(p^{(\eta+1)/3}) \quad (p \rightarrow +\infty),$$

where ω runs through the cube roots of unity. For the root near $-p$, we have

$$R_p(c) = p^{\eta-1} - 1 + \mathcal{O}(p^{-1}), \quad (p \rightarrow +\infty),$$

and, for the roots near p , we have

$$R_p(c) = 1 - p^{\eta-1} + \mathcal{O}(p^{-1/3}), \quad (p \rightarrow +\infty).$$

Since $\eta > 2/3$, it follows that $|R_p(c)| < 1$ for all c and all sufficiently large p .

(ii) Consider points of the form $p + pt$, where $t \geq 0$. A simple calculation shows that $R_p(p + pt) \geq 1$ if and only if

$$pt^3 + (2p^\eta - a)t^2 + 2(p^\eta - a)t \leq 2a.$$

For $0 \leq t \leq a/(2p^\eta)$, we have

$$pt^3 + (2p^\eta - a)t^2 + 2(p^\eta - a)t \leq a\left(\frac{p^{1-3\eta}a^2}{8}\right) + a\left(\frac{a}{2p^\eta}\right) + a,$$

which is less than $2a$ if p is sufficiently large. We conclude that, if p is sufficiently large, then $\{z \in \hat{\mathbb{C}} : |R_p(z)| \geq 1\}$ contains the interval $[p, p + ap^{1-\eta}/2]$. Therefore, the logarithmic capacity of the component of $\{z \in \hat{\mathbb{C}} : |R_p(z)| \geq 1\}$ containing p is at least as large as the capacity of $[p, p + ap^{1-\eta}/2]$, namely $ap^{\frac{1-\eta}{2}}$. \square

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